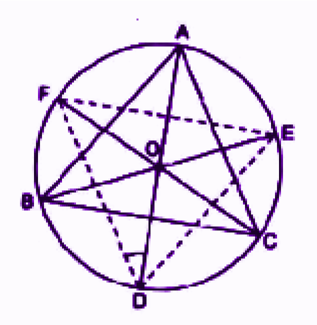
<u>**Chapter 10: Circles**</u>

Q.1.Bisectors of angles A, B and C of a triangle ABC intersect its circumcircle at D, E and F respectively. Prove that the angles of the triangle DEF are 90° – (½)A, 90° – (½)B and 90° – (½)C.

Solution:

Consider the following diagram:



Here, ABC is inscribed in a circle with center O and the bisectors of ∠A, ∠B and ∠C intersect the circumcircle at D, E and F respectively.

Now, join DE, EF and FD

As angles in the same segment are equal, so,

∠FDA = ∠FCA —————(i)

∠FDA = ∠EBA —————(i)

Adding equations (i) and (ii) we have,

∠FDA + ∠EDA = ∠FCA + ∠EBA

Or, ∠FDE = ∠FCA + ∠EBA = (½)∠C + (½)∠B

We know, ∠A + ∠B + ∠C = 180°

So, ∠FDE = (½)[∠C + ∠B] = (½)[180° – ∠A]

⇒ ∠FDE = [90 – (∠A/2)]

In a similar way,

∠FED = [90 – (∠B/2)]

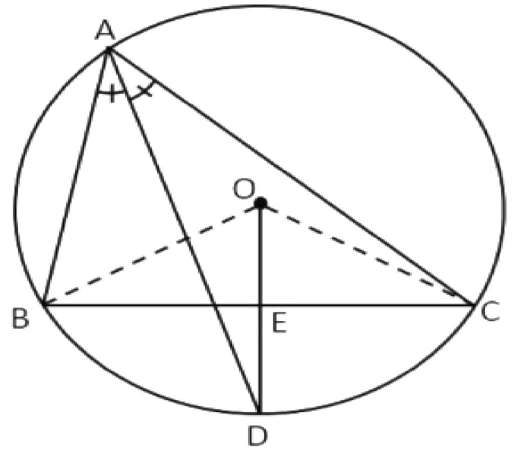
And,

∠EFD = [90 – (∠C/2)]

Q.2.In any triangle ABC, if the angle bisector of ∠A and perpendicular bisector of BC intersect, prove that they intersect on the circumcircle of the triangle ABC.

Solution:

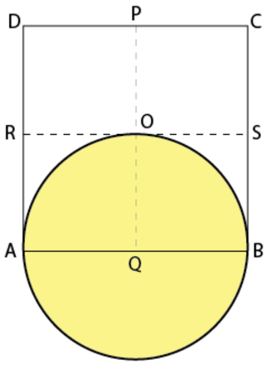
Consider this diagram:



Given: In ∆ABC, AD is the angle bisector of ∠A and OD is the perpendicular bisector of BC, intersecting each other at point D.
To Prove: D lies on the circle
Construction: Join OB and OC
Proof:
BC is a chord of the circle.
The perpendicular bisector will pass through centre O of the circumcircle.
∴ OE ⊥ BC & E is the midpoint of BC
Chord BC subtends twice the angle at the centre, as compared to any other point.
BC subtends ∠BAC on the circle & BC subtends ∠BOC on the centre
∴ ∠BAC = 1/2 ∠ BOC
In ∆ BOE and ∆COE,
BE = CE (OD bisects BC)
∠BEO = ∠CEO (Both 90°, as OD ⊥ BC)
OE = OE (Common)
∴ ∆BOE ≅ ∆COE (SAS Congruence rule)
∴ ∠BOE = ∠COE (CPCT)
Now,
∠BOC = ∠BOE + ∠COE
∠BOC = ∠BOE + ∠BOE
∠BOC = 2 ∠BOE …(2)
AD is angle bisector of ∠A
∴ ∠BAC = 2∠BAD
From (1)
∠BAC = 1/2 ∠BOC
2 ∠BAD = 1/2 (2∠BOE)
2 ∠BAD = ∠BOE
∠BAD = 1/2 ∠BOE
BD subtends ∠BOE at centre and half of its angle at Point A.
Hence, BD must be a chord.
∴ D lies on the circle.

Q.3: Prove that the circle drawn with any side of a rhombus as diameter passes through the point of intersection of its diagonals.

Solution:



To prove: A circle drawn with Q as centre, will pass through A, B and O (i.e. QA = QB = QO)

Since all sides of a rhombus are equal,

AB = DC

Now, multiply (½) on both sides

(½)AB = (½)DC

So, AQ = DP

⇒ BQ = DP

Since Q is the midpoint of AB,

AQ= BQ

Similarly,

RA = SB

Again, as PQ is drawn parallel to AD,

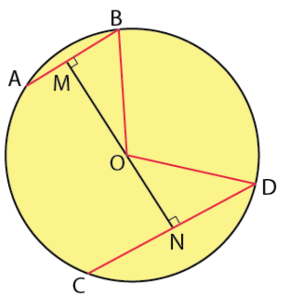
RA = QO

Now, as AQ = BQ and RA = QO we have,

QA = QB = QO (hence proved).

Q.4: Two chords AB and CD of lengths 5 cm and 11 cm respectively of a circle are parallel to each other and are on opposite sides of its centre. If the distance between AB and CD is 6, find the radius of the circle.

Solution:



Here, OM ⊥ AB and ON ⊥ CD. is drawn and OB and OD are joined.

As we know, AB bisects BM as the perpendicular from the centre bisects the chord.

Since AB = 5 so,

BM = AB/2

Similarly, ND = CD/2 = 11/2

Now, let ON be x.

So, OM = 6− x.

Consider ΔMOB,

$OB^2 = OM^2 + MB^2$

Or,

$OB^2 = 36 + x^2 – 12x + 25/4$ ……(1)

Consider ΔNOD,

$OD^2 = ON^2 + ND^2$

Or,

$OD^2 = x^2+121/4$ ……….(2)

We know, OB = OD (radii)

From eq. (1) and eq. (2) we have;

$36 + x^2 -12x + 25/4 = x^2 + 121/4$

12x = 36 + 25/4 – 121/4

12x = (144 + 25 -121)/4

12x = 48/4 = 12

x = 1

Now, from eq. (2) we have,

OD2 = 11 + (121/4)

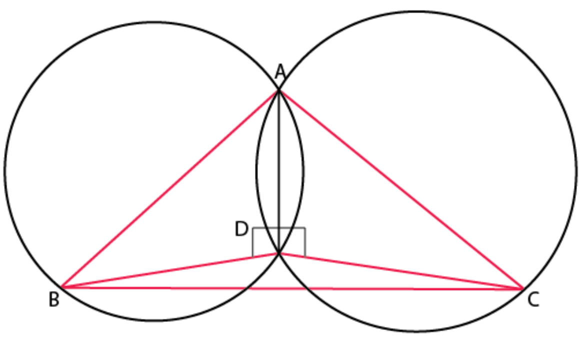
Or OD = (5/2) × √5

Q.5: If circles are drawn taking two sides of a triangle as diameters, prove that the point of intersection of these circles lies on the third side.

Solution:

First, draw a triangle ABC and then two circles having a diameter as AB and AC respectively.

We will have to now prove that D lies on BC and BDC is a straight line.



Proof:

As we know, angle in the semi-circle are equal

So, ∠ADB = ∠ADC = 90°

Hence, ∠ADB + ∠ADC = 180°

∴ ∠BDC is a straight line.

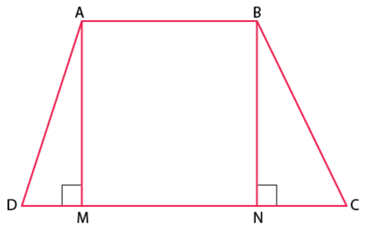
So, it can be said that D lies on the line BC.

Q.6: If the non-parallel sides of a trapezium are equal, prove that it is cyclic.

Solution:

Construction-Consider a trapezium ABCD with AB||CD and BC = AD.

Draw AM ⊥CD and BN ⊥ CD



In ∆AMD and ∆BNC;

AD = BC (Given)

∠AMD = ∠BNC (90°)

AM =BN (perpendiculars between parallel lines)

∆AMD = ∆BNC (By RHS congruency)

∆ADC = ∆BCD (By CPCT rule) …….(i)

∠BAD and ∠ADC are on the same side of transversal AD.

∠BAD + ∠ADC = 180° ……(ii)

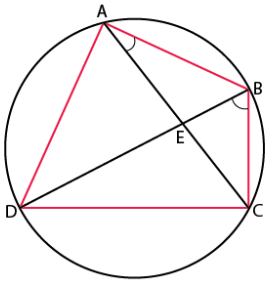
∠BAD + ∠BCD = 180° (by equation (i))

Since, the opposite angles are supplementary, therefore, ABCD is a cyclic quadrilateral.

Q.7: ABCD is a cyclic quadrilateral whose diagonals intersect at a point E. If ∠DBC = 70°, ∠BAC is 30°, find ∠BCD. Further, if AB = BC, find ∠ECD.

Solution:

Consider the following diagram.



Consider the chord CD,

As we know, angles in the same segment are equal.

So, ∠CBD = ∠CAD

∴ ∠CAD = 70°

Now, ∠BAD will be equal to the sum of angles BAC and CAD.

So, ∠BAD = ∠BAC + ∠CAD

= 30° + 70°

∴ ∠BAD = 100°

As we know, the opposite angles of a cyclic quadrilateral sum up to 180 degrees.

So,

∠BCD + ∠BAD = 180°

Since, ∠BAD = 100°

So, ∠BCD = 80°

Now consider the ΔABC.

Here, it is given that AB = BC

Also, ∠BCA = ∠CAB (Angles opposite to equal sides of a triangle)

∠BCA = 30°

also, ∠BCD = 80°

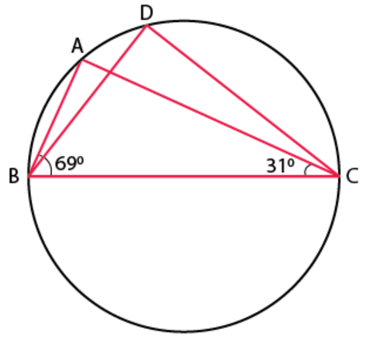
∠BCA + ∠ACD = 80°

So, ∠ACD = 50° and,

∠ECD = 50°

Q.8: In Figure, ∠ABC = 69°, ∠ ACB = 31°, find ∠BDC.

Solution:



As we know, angles in the segment of the circle are equal so,

∠BAC = ∠BDC

Now in the In ΔABC, sum of all the interior angles will be 180°

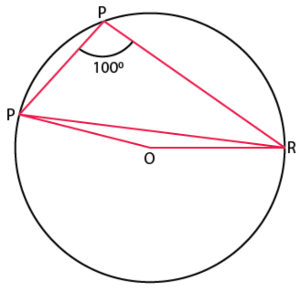
So, ∠ABC + ∠BAC + ∠ACB = 180°

Now, by putting the values,

∠BAC = 180° – 69° – 31°

So, ∠BAC = 80°

Q.9: In Figure, ∠PQR = 100°, where P, Q and R are points on a circle with centre O. Find ∠OPR.



Solution:

Since angle which is subtended by an arc at the centre of the circle is double the angle subtended by that arc at any point on the remaining part of the circle.

So, the reflex ∠POR = 2 × ∠PQR

We know the values of angle PQR as 100°

So, ∠POR = 2 × 100° = 200°

∴ ∠POR = 360° – 200° = 160°

Now, in ΔOPR,

OP and OR are the radii of the circle

So, OP = OR

Also, ∠OPR = ∠ORP

Now, we know sum of the angles in a triangle is equal to 180 degrees

So,

∠POR + ∠OPR + ∠ORP = 180°

⇒ ∠OPR + ∠OPR = 180° – 160°

As ∠OPR = ∠ORP

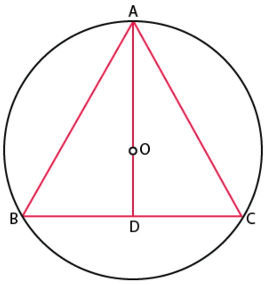
⇒ 2∠OPR = 20°

Thus, ∠OPR = 10°

Q.10: A circular park of radius 20m is situated in a colony. Three boys Ankur, Syed and David are sitting at equal distance on its boundary each having a toy telephone in his hands to talk each other. Find the length of the string of each phone.

Solution:

First, draw a diagram according to the given statements. The diagram will look as follows.



Here the positions of Ankur, Syed and David are represented as A, B and C respectively. Since they are sitting at equal distances, the triangle ABC will form an equilateral triangle.

AD ⊥ BC is drawn. Now, AD is median of ΔABC and it passes through the centre O.

Also, O is the centroid of the ΔABC. OA is the radius of the triangle.

OA = 2/3 AD

Let the side of a triangle a metres then BD = a/2 m.

Applying Pythagoras theorem in ΔABD,

$AB^2 = BD^2 + AD^2$

$\Rightarrow AD^2 = AB^2 – BD^2$

$\Rightarrow AD^2 = a^2 – (a/2)^2$

$\Rightarrow AD^2 = 3a2/4$

$\Rightarrow AD = \sqrt{3}a/2$

OA = 2/3 AD

$\Rightarrow 20 \text{ m} = 2/3 \times \sqrt{3}a/2$

$\Rightarrow a = 20\sqrt{3} \text{ m}$

So, the length of the string of the toy is $20\sqrt{3}$ m.

Q.11: If two equal chords of a circle intersect within the circle, prove that the line joining the point of intersection to the centre makes equal angles with the chords.

Solution:

From the question we have the following conditions:

(i) AB and CD are 2 chords which are intersecting at point E.

(ii) PQ is the diameter of the circle.

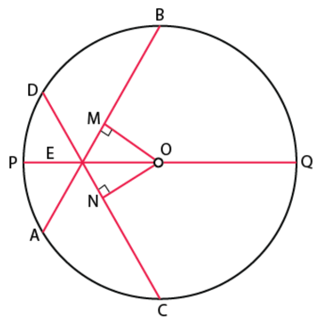
(iii) AB = CD.

Now, we will have to prove that ∠BEQ = ∠CEQ

For this, the following construction has to be done:

Construction:

Draw two perpendiculars are drawn as OM ⊥ AB and ON ⊥ CD. Now, join OE. The constructed diagram will look as follows:



Now, consider the triangles ΔOEM and ΔOEN.

Here,

(i) OM = ON [Since the equal chords are always equidistant from the centre]

(ii) OE = OE [It is the common side]

(iii) ∠OME = ∠ONE [These are the perpendiculars]

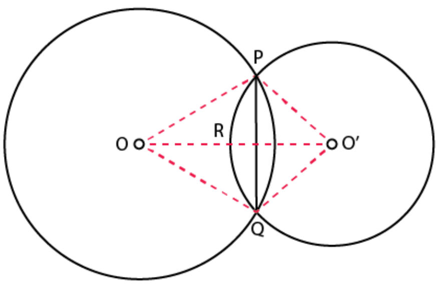
So, by RHS similarity criterion, ΔOEM ≅ ΔOEN.

Hence, by CPCT rule, ∠MEO = ∠NEO

∴ ∠BEQ = ∠CEQ (Hence proved).

Q.12: If two circles intersect at two points, prove that their centres lie on the perpendicular bisector of the common chord.

Solution:



It is given that two circles intersect each other at P and Q.

To prove:

OO’ is a perpendicular bisector of PQ.

Proof:

Triangle ΔPOO’ and ΔQOO’ are similar by SSS congruency since

OP = OQ and O’P = O’Q (Since they are also the radii)

OO’ = OO’ (It is the common side)

So, It can be said that ΔPOO’ ≅ ΔQOO’

∴ ∠POO’ = ∠QOO’ — (i)

Even triangles ΔPOR and ΔQOR are similar by SAS congruency as

OP = OQ (Radii)

∠POR = ∠QOR (As ∠POO' = ∠QOO')

OR = OR (Common arm)

So, ΔPOR ≅ ΔQOR

∴ ∠PRO = ∠QRO

Also, As we know,

∠PRO + ∠QRO = 180°

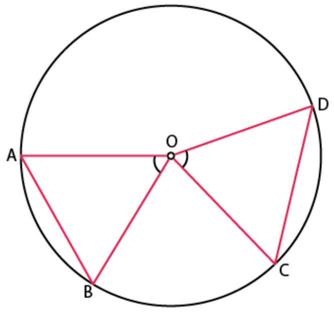
Hence, ∠PRO = ∠QRO = 180°/2 = 90°

So, OO' is the perpendicular bisector of PQ.

Q.13: Prove that if chords of congruent circles subtend equal angles at their centres, then the chords are equal.

Solution:

Consider the following diagram-



Here, it is given that ∠AOB = ∠COD i.e. they are equal angles.

Now, we will have to prove that the line segments AB and CD are equal i.e. AB = CD.

Proof:

In triangles AOB and COD,

∠AOB = ∠COD (as given in the question)

OA = OC and OB = OD ((these are the radii of the circle)

So, by SAS congruency, ΔAOB ≅ ΔCOD.

∴ By the rule of CPCT, AB = CD. (Hence proved).